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G-invariant polynomial extensions of Lie algebras in quantum many-body physics

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Abstract. A new class of Lie-algebraic structures g_d is revealed in some multi-particle processes of quantum physics having internal symmetry groups G_{inv}. They are extensions of some Lie algebras h (via coset construction) by G-invariant h-tensors v which are polynomials in boson operators. Applications of these algebras are briefly discussed for solving spectral and evolution tasks with g_d as their dynamic symmetry algebras.

1. Introduction

The mathematical apparatus of Lie groups and algebras has been widely and fruitfully exploited in different branches of quantum physics from the time of its origin [1–3]. Specifically, it provides powerful techniques (Wigner–Racah algebras [2, 4], generalized coherent states (GCS) [5, 6] etc.) for solving both spectral and evolution tasks with Hamiltonians H^0 given by quadratic forms in the second quantization operators a_j^+, a_j which are easily transformed in linear combinations of Lie algebras generators with the help of the Jordan–Schwinger (JS) mapping [3, 4, 7]. But many Hamiltonians of quantum many-body physics have no such simple form (see, e.g., [8–15] and references therein) and may be represented at best by certain elements of universal enveloping algebras $U(g)$ of some Lie algebras 'g'. In certain special models such nonlinear Hamiltonians can be transformed to linear forms in generators of some Lie algebras via the Holstein–Primakoff (HP) type mappings [13, 14] or their extensions [9, 11, 12, 15]. But, in general, it is not the case, and therefore a direct application of Lie-algebraic techniques to solving physical tasks is less efficient than for linear realizations of Hamiltonians in generators of Lie algebras.

But recently some new Lie-algebraic structures (quantum groups or algebras, W-algebras etc. [16–20]) have been introduced in different areas of modern physics. All these objects, called nonlinear or deformed Lie algebras 'g_d' [20], are generated by certain sets $B=\{T_c\}$ of their generators T_c satisfying commutation relations (CR) of the form

$$[T_a, T_b]=f_{ab}(\{T_c\}) \tag{1.1}$$

where $f_{ab}(\ldots.)$ are functions of the generators T_c given by power series and constrained by the Jacobi identities (which are fulfilled automatically when sets B are embedded in associative algebras $A(B)$). Many deformed algebras g_d used in physics have the so-called coset structure (see [19, 20])

$$g_d = h + v \tag{1.2}$$

and may be viewed as extensions of usual Lie algebras $h = \{E_a\}$ by some h-tensor operators $v = \{V_c\}$ satisfying the CRS

$$[E_a, V_b] = \sum_c \tau^c_{ab} V_c, \tag{1.3a}$$

$$[V_a, V_b] = f_{ab}(E_c) \qquad V_c \in v, E_a \in h \tag{1.3b}$$

where τ^c_{ab} are matrix elements of operators V_c and $f_{ab}(\ldots)$ are power series in generators of the subalgebra h only. Evidently, this construction generalizes the well-known one for oscillator algebras $\mathrm{osc}(m) = u(m) \oplus h(m)$ [6] and the Cartan expansion for usual real semi-simple Lie algebras [21].

Until quite recently such deformed Lie algebras g_d were examined mainly in the context of quantum field theory and statistical physics models [16–20]. But results of recent papers [22–29] show certain possibilities of their applications in other branches of modern quantum physics. Specifically, in [22–25] it was shown that deformed Lie algebras g_d with the structure (1.2)–(1.3) arise in a natural manner in some composite many-body physics models with Hamiltonians H having some symmetry groups G_{inv} ($[H, G_{inv}] = 0$) and presented as linear forms in elements of finite sets $I(G_{inv})$ of basic invariants of groups G_{inv}. The sets $I(G_{inv})$ endowed by commutators $[A, B] = AB - BA$ lead, in general, to deformed Lie algebras g_d of the (1.2) type which retain some properties of usual Lie algebras and, besides, form together with G_{inv} generalized Weyl-Howe's dual pairs ($D_1 = G_{inv}$, $D_2 = g_d$) which act complementarily [30, 23] on the Hilbert spaces $L(H)$ of quantum states of models under study. All this opens up possibilities of application of the g_d formalism to solving physical problems by analogy with the usual Lie algebras (cf [6, 15]).

The aim of this paper is to show the efficiency of the approach [22–25] in quantum many-body physics on simple models.

In section 2 we display deformed Lie algebras $g_d(H)$ as algebras of dynamic symmetry (DS) in typical and widespread models whose Hamiltonians H are invariant with respect to certain groups $G_{inv}(H)$ and outline their representations on the model spaces $L(H)$. A more complete list of such models can be found in the papers [25]. In section 3 we consider some applications of the $g_d(H)$ formalism to solving spectral and evolution tasks. In section 4 we discuss some generalizations of the results obtained. In the appendix we consider the example of the three-boson model.

2. Polynomial deformations of Lie algebras in many-particle processes exhibiting some G_{inv}-symmetries

Let us consider quantum many-body models with essentially nonlinear multiboson Hamiltonians of the form

$$H = \sum_{i,j=1}^{m_1} \omega_{ij} a_i^+ a_j + \sum_{i,j=1}^{m_2} \tilde{\omega}_{ij} b_i^+ b_j + \sum_{\{i_\alpha, j_\beta\}} [g^{j_1 \ldots j_p}_{i_1 \ldots i_n} a^+_{i_1} \ldots a^+_{i_n} b_{j_1} \ldots b_{j_p} + \mathrm{HC}] \tag{2.1}$$

where $g\ldots$ are constants or time-dependent functions, a_i^+, a_j, b_i^+, b_j are boson operators, and non-quadratic parts of H describe different scattering processes, including creation/absorption of multiboson clusters in external classical (for $m_1 = 0$ or

$m_2=0$) and quantized fields [6, 8, 10–12, 15], as well as n-photon squeezing and frequency-conversion in quantum optics [9, 25].

By a direct inspection of (2.1) we find the symmetry group $G_{inv}(H)$ of the Hamiltonian H: $C_n \otimes C_p \otimes G^{int}_{inv}$ where $C_n = \{c_{kn} = \exp(i2\pi k/n)$: $a_i^+ \to c_{kn} a_i^+$, $k=0, 1, \ldots, n-1\}$; $C_p = \{c_{kp}$: $b_i^+ \to c_{kp} b_i^+$, $k=0, 1, \ldots, p-1\}$ and $G^{int}_{inv} = \exp(g^{int}_{inv}) = \exp(\Sigma \lambda_\alpha R_\alpha)$. We note that continious subgroups G^{int}_{inv} characterize symmetry properties of the interaction of a- and b-boson subsystems; they are generated by the integral of motion

$$R_0 = p \sum_i a_i^+ a_i + n \sum_j b_j^+ b_j$$

and, perhaps, some other

$$R_\alpha = \sum_i \mu_i^\alpha a_i^+ a_i + \sum_i \eta_i^\alpha b_i^+ b_i, \alpha = 1, \ldots, k$$

whose number k depends on concrete forms of Hamiltonians (2.1) (see [25] and the appendix of this paper). Appropriate sets $B = I(G_{inv})$ of basic invariants composed of the second quantization operators a_i^+, a_j, b_i^+, b_j are given as follows [25]:

$$B = B^h \oplus B^v \tag{2.2a}$$

$$B^h = \{E_{ij} = a_i^+ a_j, i, j = 1, \ldots, m_1\} \cup \{\tilde{E}_{ij} = b_i^+ b_j, i, j = 1, \ldots, m_2\} \tag{2.2b}$$

$$B^v = \{Y^{j_1 \cdots j_p}_{i_1 \cdots i_n} = a^+_{i_1} \ldots a^+_{i_n} b_{j_1} \ldots b_{j_p}\} \cup \{(Y^{j_1 \cdots j_p}_{i_1 \cdots i_n})^+\} \tag{2.2c}$$

where B^h corresponds to the quadratic part H^0 of H and generates (via the standard *JS* mapping) usual unitary Lie algebras $h(B) = u(m_1) \oplus u(m_2)$, $u(m_1) = \mathrm{Span}\{E_{ij}\}$, $u(m_2) = \mathrm{Span}\{\tilde{E}_{ij}\}$ while B^v generates the coset space $v(B) = \mathrm{Span}\ \{Y^{j_1 \cdots j_p}_{i_1 \cdots i_n}, (Y^{j_1 \cdots j_p}_{i_1 \cdots i_n})^+\}$ of the $(u(m_1) \oplus u(m_2))$-symmetric tensors which, evidently, obey CRs of the (1.3a) type,

$$[E_{ij}, Y^{j_1 \cdots j_p}_{i_1 \cdots i_n}] = \delta_{ji_1} Y^{j_1 \cdots j_p}_{ii_2 \ldots i_n} + \delta_{ji_2} Y^{j_1 \cdots j_p}_{i_1 i i_3 \ldots i_n} + \cdots \tag{2.3a}$$

$$[\tilde{E}_{ij}, Y^{j_1 \cdots j_p}_{i_1 \cdots i_n}] = -(\delta_{ij_1} Y^{jj_2 \cdots j_p}_{i_1 \cdots i_n} + \delta_{ij_2} Y^{j_1 j j_3 \cdots j_p}_{i_1 \cdots i_n} + \cdots). \tag{2.3b}$$

At the same time elements of B satisfy some extra polynomial relations (syzygies) which follow from their specific forms as G_{inv}-invariants and the CRs for a_i^+, a_j, b_i^+, b_j [24, 25]. Specifically, from (2.2) we find

$$\begin{aligned} Y^{j_1 \cdots j_p}_{i_1 \cdots i_n} (Y^{s_1 \cdots s_p}_{t_1 \cdots t_n})^+ &= A^{j_1 \cdots j_p; s_1 \cdots s_p}_{i_1 \cdots i_n; t_1 \cdots t_n}(\{E_{ij}; \tilde{E}_{ij}\}) \\ (Y^{s_1 \cdots s_p}_{t_1 \cdots t_n})^+ Y^{j_1 \cdots j_p}_{i_1 \cdots i_n} &= B^{j_1 \cdots j_p; s_1 \cdots s_p}_{i_1 \cdots i_n; t_1 \cdots t_p}(\{E_{ij}; \tilde{E}_{ij}\}) \end{aligned} \tag{2.4}$$

where $A^{\cdots}_{\cdots}(\ldots)$ and $B^{\cdots}_{\cdots}(\ldots)$ are polynomials of the nth order in variables E_{ij} and of the pth order in variables $\tilde{E}_{ij}$ (with the same leading terms for $A^{\cdots}_{\cdots}(\ldots)$ and $B^{\cdots}_{\cdots}(\ldots)$); for example, in the case $m_2=0$ we have

$$Y_{i \ldots i}(Y_{i \ldots i})^+ = \prod_{l=0}^{n-1} (E_{ii} - l) \equiv (E_{ii})^{(n)}$$

$$(Y_{i \ldots i})^+ Y_{i \ldots i} = (E_{ii} + n)^{(n)}.$$

From (2.2) and (2.4) one gets immediately CRs of the (1.3b) type for the coset generators

$$\begin{aligned} [Y^{j_1 \cdots j_p}_{i_1 \cdots i_n}, Y^{s_1 \cdots s_p}_{t_1 \cdots t_n}] &= 0 = [(Y^{j_1 \cdots j_p}_{i_1 \cdots i_n})^+, (Y^{s_1 \cdots s_p}_{t_1 \cdots t_n})^+] \\ [(Y^{j_1 \cdots j_p}_{i_1 \cdots i_n})^+, Y^{s_1 \cdots s_p}_{t_1 \cdots t_n}] &= P^{j_1 \cdots j_p; s_1 \cdots s_p}_{i_1 \cdots i_n; t_1 \cdots t_n}(\{E_{ij}; \tilde{E}_{ij}\}) \end{aligned} \tag{2.5}$$

where $P^{\cdots}_{\cdots}(\ldots)$ are polynomials of the $(n+p-1)$th order in E_{ij}, $\tilde{E}_{ij}$.

Thus, extending the $(u(m_1)\oplus u(m_2))/g^{\text{int}}_{\text{inv}}$ algebras (as quadratic DS subalgebras of H) by the $G_{\text{inv}}(H)$-invariant $(u(m_1)\oplus u(m_2))$ – tensors embedded in $U(h(m_1)\oplus h(m_2))$ we have obtained some new Lie-algebraic structures $g_d(H)$ of the (1.2), (1.3) type (polynomial Lie algebras) as full DS algebras of H. Evidently, this way of introducing coherent dynamic variables $\in g_d(H)$ generalizes the JS mapping and, therefore, may be named as the G-invariant polynomial JS mapping. In their properties algebras $g_d(H)$ are more similar to W-algebras [17] than q-algebras since they have no continuous deformation parameters like q. At the same time the numbers 'n', 'p' can be considered as specific discrete deformation parameters. Therefore algebras $g_d(H)$ may be viewed as specific (polynomial) deformations $\mathrm{osc}_d^{(n;p)}(m)$, $m=m_1+m_2$, of the usual oscillator algebras osc(m) (cf [31]) since they are reduced to osc(m) for lowest values of n, p: $n+p=1$; the label $(n;p)$ indicates that coset generators are $(u(m_1)\oplus u(m_2))$-symmetric tensors [25]. Besides, algebras $g_d(H)$ may be considered as deformations $sp_d^{(n)}(2m;R)$ (for $p=0$) or $su_d^{(n;p)}(m)$ (for $n\neq 0$, $p\neq 0$) of Lie algebras $sp(2m;R)$ or $su(m)$ since $g_d(H)=sp(2m;R)$ for $n=2$, $p=0$ and $g_d(H)=su(m)$ for $n=1, p=1$. So, we have got two different kinds of polynomial Lie algebras g_d related to compact or non-compact Lie algebras, depending on the occurrence or the absence of coupling a- and b-subsystems.

As was mentioned, algebras $g_d(H)$ form together with groups $G_{\text{inv}}(H)$ generalized Weyl–Howe pairs on the spaces

$$L(H)=L_F(m)=\mathrm{Span}\{|n_\alpha, n_\beta\rangle=\prod_{\alpha,\beta}(a_\alpha^+)^{n_\alpha}(b_\beta^+)^{n_\beta}|0\rangle\}$$

that leads to the decompositions

$$L_F(m)=\sum_{[l_i]}\oplus L([l_i]) \tag{2.6}$$

of $L_F(m)$ into direct sums of the subspaces $L([l_i])$ which are invariant with respect to actions of both $G_{\text{inv}}(H)$ and $g_d(H)$ (the label $[l_i]$ specifies simultaneously irreducible representations (irreps) of both G_{inv} and g_d). Besides, algebras $g_d(H)$ resemble in their structure properties real semi-simple algebras, and, therefore, algebraic properties and representations of the first ones can be developed in parallel to those of the latter ones [20, 25].

For further clarification of the above remarks we consider in more detail the simplest example when $m_1=1$, $m_2=1$ (see also the appendix). Introducing the notation $Y_0=(E_{11}-\bar{E}_{11})/(n+p)$, $Y_+=Y^{1;\cdots;1}_{1;\cdots;1}$, $Y_-=(Y_+)^+$ we find from (2.3)–(2.5) the CRS

$$[Y_0, Y_\pm]=\pm Y_\pm \tag{2.7a}$$

$$\begin{aligned}&[Y_-, Y_+]=\varphi_{n,p}(Y_0)=\psi_{n,p}(Y_0+1)-\psi_{n,p}(Y_0)\\&\psi_{n,p}(Y_0)=(nY_0+R)^{(n)}(R-p(Y_0-1))^{(p)}\\&R=R_0=(pE_{11}+n\tilde{E}_{11})/(n+p)\\&[R, Y_\alpha]=0 \qquad (a)^{(b)}=a(a-1)\ldots(a-b+1)\end{aligned} \tag{2.7b}$$

that allows us to identify $g_d(H)$ as deformations $sl_d^{(n;p)}(2)$ of the Lie algebra $sl(2)$. In accordance with the above general remarks we will distinguish compact ($sl_d^{(n;p)}(2)=su_d^{(n;p)}(2)$, p, $n\neq 0$) and non-compact ($sl_d^{(n;0)}(2)=su_d^{(n)}(1,1)$, $p=0$, $n\neq 0$) versions of $sl_d^{(n;p)}(2)$.

It is easy to check that algebras $sl_d^{(n;p)}(2)$ have the Casimir operators $C_d^{(n;p)}(2)$ given by the expressions

$$C_2^{(n;p)}(2)=-Y_+Y_-+\psi_{n,p}(Y_0) \tag{2.8}$$

that is a deformation of the Casimir operator $C_2(2)=\pm E_+E_-+E_0^{(2)}$ of the algebra $sl(2)$ [21]. This relationship of algebras $sl_d^{(n;p)}(2)$ with the usual Lie algebras allows us to develop a theory of representations of the first ones by analogy with that of the usual Lie algebras [20].

Specifically, using the above realization (2.2) we can determine all irreps of $sl_d^{(n;p)}(2)$ which act on $L(H)=L_F(m)$. Namely, because of the identities (2.4) we find that $C_2^{(n;p)}(2)_B=0$ on $L(H)$ where the subscript 'B' denotes the boson realization. From here it follows immediately that $su_d^{(n)}(1,1)$ has on $L_F(1)$ only n infinite-dimensional irreps $D(l_1)$ specified by the lowest weights $l_1=k/n$, $k=0,1,\ldots,n=1$, and the lowest vectors $|l_1\rangle=[k!]^{-1/2}(a_1^+)^k|0\rangle\,(Y_0|l_1\rangle=l_1|l_1\rangle$, $Y_-|l_1\rangle=0)$. At the same time the algebra $su_d^{(n;p)}(2)$ has on $L(H)=L_F(2)$ an infinite number of finite-dimensional irreps $D(l_1,l_2)$ (with dimensions $d(l_1,l_2)=[[s/p]]+1=[[l_2/p-l_1]]+1$) which are specified by the lowest weights $l_1=(k-s)/(n+p)$, eigenvalues $l_2=(kp+ns)/(n+p)$ of the above operator R in (2.7), $k=0,1,\ldots,n-1$, $s=0,1,\ldots,$ and the lowest vectors $|l_1,l_2\rangle=[k!s!]^{-1/2}(a_1^+)^k(b_1^+)^s|0\rangle\,(Y_0|[l_i]\rangle=l_1|[l_i]\rangle$, $R|[l_i]\rangle=l_2|[l_i]\rangle$, $Y_-|[l_i]\rangle=0)$; here $[[x]]$ is the integral part of the number 'x'. All other basic vectors $|[l_i];t\rangle$, $t>0$, of the $sl_d^{(n;p)}(2)$ irreps have the form

$$|[l_i];t\rangle=N([l_i],t)(Y_+)^t|[l_i]\rangle$$

where

$$(N(\ldots))^{-2}=\langle[l_i]|(Y_-)^t(Y_+)^t|[l_i]\rangle=\prod_{r=0}^{t-1}\psi_{n,p}(l_1+t-r)\equiv[\psi_{n,p}(l_1+t)]^{(t)}.$$

Without dwelling on other features of the $sl_d^{(n,p)}(2)$ irreps and their generalizations for $g_d(H)$ with any m_α (see [20, 21–25]) we note only that the $su_d^{(n)}(1,1)$ GCS $|\alpha;[l_1]\rangle=\exp(\alpha Y_+-\alpha^*Y_-)|[l_1]\rangle$ of the group orbit type are not analytical vectors for all values $n>2$ [32] unlike the GCS which are eigenfunctions of the lowering operator Y_- [25].

3. Polynomial deformations of $sl_\alpha^{(n;p)}(2)$ in solving physical problems

By analogy with the case $n\leqslant 2$ [6] one can expect that the theory of deformed Lie algebras may be useful for treating different problems with Hamiltonians (2.1). Specifically, the above g_d-invariant decompositions of $L(H)$ of the (2.6) type allow us to examine models under study independently on each g_d-invariant subspace $L([l_i])$. But for lack of simple formulas for disentangling exponents of the g_d elements [32] one cannot apply their orbit GCS technique for diagonalizing H or for finding appropriate evolution operators $U_H(t;t_0)$, as is the case for familiar Lie algebras [5, 6]. Nevertheless, below we show that there exist some possibilities of the g_d formalism application to solving these tasks. We focus on the case of the $sl_d^{(n;p)}(2)$ algebra, since this is by far the simplest one, allowing elucidation of the main ideas of our analysis.

One way of applying the g_d formalism is connected with finding bound states of the stationary Schrödinger equation

$$H|E_\alpha\rangle = E_\alpha|E_\alpha\rangle \tag{3.1}$$

with the Hamiltonians (2.1) on the invariant subspaces $L([l_i]) \subset L(H)$. In the case $g_d = sl_d^{(n,p)}(2)$ one can present H from (2.1) in the form

$$H = aY_0 + bY_+ + b^*Y_- + C \qquad [Y_\alpha, C] = 0. \tag{3.2}$$

We will seek solutions of (3.1) in the form [25]

$$|E\rangle = \sum_f Q_f(E)(Y_+)^f|[l_i]\rangle \tag{3.3}$$

that corresponds to the diagonalization scheme [33] of any elements of the usual Lie algebra $sl(2)$. When substituting (3.3) into (3.1) and using (3.2) and (2.6) one obtains the recurrence relations (finite-difference equations)

$$[(l_1+f)a-\lambda]Q_f + bQ_{f-1} + b^*\psi_{n,p}(l_1+f+1)Q_{f+1} = 0 \qquad f=0,1,\ldots \tag{3.4}$$

for determining the coefficients $Q_f = Q_f(E)$; here $\psi_{n,p}(x)$ is the $sl_d^{(n,p)}(2)$ structure polynomial from (2.7) and the spectral parameter $\lambda = E - c$ comprises both the energy eigenvalue and that of the invariant operator C which is constant on the whole $L([l_i])$.

Difference equations (3.4) belong to the hypergeometric type [34] and for structure polynomials $\psi_{n,p}(x)$, $n+p=2$, related to the usual $sl(2)$ their solutions are expressed in terms of classical orthogonal polynomials in the discrete variable λ [33, 34]. In the case of deformed algebras $sl_d^{(n,\epsilon)}(2)$ we get in such a manner new classes of orthogonal polynomials in a discrete variable $\lambda(E)$ on inhomogeneous (in general) lattices related to non-equidistant energy spectra of H in (2.1). The concrete forms of these lattices and spectra can be found by solving characteristic equations

$$F_{l_1}(\lambda) = 0 \tag{3.5}$$

where $F_{l_1}(\lambda)$ are determinants of tridiagonal matrices of coefficients in (3.4). For the non-compact version $su_d^{(n)}(1,1)$ of $sl_d^{(n,0)}(2)$, when all subspaces $L([l_1])$ are infinite-dimensional, from (3.4) one easily obtain recurrence relations (resembling those for the Bessel functions $J_k(\lambda)$ [34])

$$[a(k-1)-\lambda]F_k(\lambda) = F_{k-1}(\lambda) + |b|^2\psi_{n,p}(k)F_{k+1}(\lambda) \tag{3.6}$$

for determining $F_k(\lambda)$, while for the compact version $su_d^{(n,p)}(2)$ this is not the case. However, for $su_d^{(n;p)}(2)$ all irreps on $L([l_i])$ are two-side bounded and one can use, instead of (3.5), the condition

$$Q_d(E) = 0 \Rightarrow [(l_1+d-1)a-\lambda]Q_{d-1} + bQ_{d-2} = 0 \tag{3.7}$$

for determining spectra $\{E_\alpha = E_\alpha(l_i)\}$ where $d = d(l_1, l_2)$ is the $L([l_1, l_2])$ dimension.

So, solving (3.4)–(3.7) together with appropriate boundary conditions, we find eigenfunctions and energy spectra of Hamiltonians (3.2) that allow us to examine and display the dynamic features of models under study [9, 28, 35, 36]. In particular, in the case of $su_d^{(n;p)}(2)$ equations (3.4) and (3.7) are easily solved by hand for $L([l_1l_2])$ of lowest dimensions. An inspection of the form of these solutions allows us to determine periodic or almost periodic regimes of time evolution of models under study for certain initial conditions and specific values of coupling constants [35, 36].

One way of solving (3.4) is provided by the generating function method, when instead of the coefficients $Q_f(E)$ one seeks generating functions

$$u(z;E)=\sum_f Q_f(E)z^f. \tag{3.8}$$

These functions are related to $|E\rangle$ by the equation

$$|E\rangle=u(Y_+;E)|[l_i]\rangle \tag{3.9}$$

and satisfy differential equations of the Fuchs type [34]

$$\{al_1-\lambda+az\,\mathrm{d}/\mathrm{d}z+b^*z^{-1}\psi_{n,p}(z\mathrm{d}/\mathrm{d}z+l_1)\}u(z;E)=0 \tag{3.10}$$

which are obtained from (3.4) and (3.8) with the help of equations

$$\psi_{n,p}(l_1+x)=\sum_s c_s x^{(s)} \qquad c_s=(s!)^{-1}\Delta_x^s\psi_{n,p}(l_1+x)|_{x=0}$$
$$\Delta_x f(x)=f(x+1)-f(x) \tag{3.11a}$$

$$(z\mathrm{d}/\mathrm{d}z)^{(s)}=z^s\mathrm{d}^s/\mathrm{d}z^s \qquad z\mathrm{d}/\mathrm{d}z z^f=fz^f \qquad A^{(B)}=A(A-1)\ldots(A-B+1) \tag{3.11b}$$

and are related to the realization

$$Y_+=z \qquad Y_0=(z\mathrm{d}/\mathrm{d}z+l_1) \qquad Y_-=z^{-1}\psi_{n,p}(z\mathrm{d}/\mathrm{d}z+l_1)=\sum_{s=1}^{n+p}c_s z^{s-1}\mathrm{d}^s/\mathrm{d}z^s \tag{3.12}$$

of the $sl_d^{(n,p)}(2)$ generators Y_α by means of differential operators. We note that (3.10) are close in their form to those defining generalized hypergeometric functions ${}_pF_q(\ldots;z)$ in the continuous variable 'z' ([34], vol 1, ch 4); besides, because of (3.1), (3.8) and (3.9) their solutions seem to give some new (non-classical) classes of orthogonal polynomials in 'z'.

So, using the $sl_d^{(n,p)}(2)$ formalism, one can obtain solutions of the (3.1) and (3.2) with the help of solving either finite-difference equations (3.4) or differential equations (3.10) of special types. We also note that from (3.10) one can find its non-stationary analogue

$$\mathrm{i}\hbar\partial u(z;t)/\partial t=\{a(l_1+\mathrm{d}/\mathrm{d}z)+bz+z^{-1}b^*\psi_{n,p}(z\mathrm{d}/\mathrm{d}z+l_1)\}u(z;t) \tag{3.13}$$

defining solutions of an appropriate non-stationary Schrödinger equation related to (3.1). We point out that in general (3.13) contain higher derivatives $\mathrm{d}^s u/\mathrm{d}z^s$ and their solutions (expressed as contour or multiple integrals) are singular [34].

Another way of appliying the $sl_d^{(n;p)}(2)$ formalism to solve physical tasks is based on comparing Hamiltonians (3.2) with 'distorted' Hamiltonians

$$H^D=\tilde{a}V_0+\tilde{b}V_++\tilde{b}^*V_-+\tilde{C} \qquad [\tilde{C},V_\alpha]=0 \tag{3.14}$$

which are linear in generators V_α of certain usual Lie algebras $g(H^D)$. Herewith, algebras $g(H^D)$ are related to $sl_d^{(n,p)}(2)$ via the HP-type mapping

$$sl_d^{(n;p)}(2)\to g(H^D)=\{V_\alpha,\alpha=0,\pm:[V_0,V_\pm]=\pm V_\pm,[V_-,V_+]=\alpha V_0+\beta\} \tag{3.15a}$$

$$V_0=Y_0+\mu \qquad V_+=Y_+f(Y_0) \qquad V_-=f(Y_0)Y_- \tag{3.15b}$$

where functions $f(Y_0)$ are determined from the equations [37]

$$\phi(Y_0)-\phi(Y_0-1)=\alpha Y_0+\alpha\mu+\beta \qquad \phi(Y_0)=|f(Y_0)|^2\psi_{n,p}(Y_0+1) \tag{3.16}$$

with $\psi_{n,p}(Y_0)$ being the structure polynomial in (2.7*b*).

For example, taking $g(H^D)=\mathrm{osc}(1)$ (with $\alpha=0$, $\beta=1$ in (3.15*a*)), we find

$$|f(Y_0)|^2=(Y_0+1+\lambda)/\psi_{n,p}(Y_0+1) \tag{3.17}$$

whereas for $g(H^D)=su(2)/su(1,1)$ (with $\alpha=\mp 2$, $\beta=0$) one obtains

$$|f(Y_0)|^2=[\lambda-(Y_0+1+\mu)]/\psi_{n,p}(Y_0+1). \tag{3.18}$$

Herewith constants λ, μ in (3.17) and (3.18) are determined from conditions of a canonical behaviour of operators V_α of the $g(H^D)$ irreps realized on the subspaces $L([l_i])$ [25, 37].

Thus, generalized HP mappings (3.15) allow us to compare original nonlinear problems with their specific linearized versions governed by distorted Hamiltonians (3.14) and admitting exact solutions with the help of the Lie-algebraic techniques [4–6, 15]. Solutions of such linearized models can be viewed (at an adequate choice of parameters a, b in (3.14)) as specific smooth (analytical) approximations modulating exact (generally speaking, non-analytical [32]) solutions of models with Hamiltonians (3.2); herewith one may use relative moments [38]

$$\delta_p(H,H^D)=|\mathrm{Tr}(H-H^D)^p/\mathrm{Tr}(H)^p| \qquad p=1,2,\ldots \tag{3.19}$$

(with traces taken over invariant subspaces $L([l_i])\subset L(H)$ or whole spaces $L(H)$) as proximity measures of such approximations. These linearized solutions can be also used as specific zeroth-order approximations for developing perturbative and other iterative schemes of solving original tasks [27, 35] which are related to finding approximate solutions of (3.4)–(3.7), (3.10) and (3.13). Examples of such developments will be given in forthcoming papers.

4. Conclusion

In conclusion we outline some possible developments of the results above. One line of investigations is concerned with extensions of $u(m)$ in models of the (2.1) type by using other than symmetric tensors $V_c(\{a_i^+, a_j\})$ in their boson or fermion realization as coset generators in (1.2); for example, such extensions by skew-symmetric tensors are relevant as DS algebras to composite models with the internal $SU(n)$ symmetries [24]. This opens a way for obtaining nonlinear models in quantum physics which are solvable by techniques sketched in the previous section [22–25]. Another generalization here is related to extensions of other Lie algebras $h\neq u(m)$ (as DS algebras of models with quadratic Hamiltonians) via the scheme (1.2). For example, in models with $G_{\mathrm{inv}}=SO(n)$ we get $h=sp(2m,R)$ [30] and

$$v=\mathrm{Span}\{[a_{i_1}^+\ldots a_{i_k}^+a_{j_1}\ldots a_{j_{n-k}}]=\sum\varepsilon^{\alpha_1\alpha_2\cdots\alpha_n}a_{i_1\alpha_1}^+\ldots a_{i_k\alpha_k}^+a_{j_1\alpha_{k+1}}\ldots a_{j_{n-2}\alpha_n};$$

$$\varepsilon^{\alpha_1\cdots\alpha_n}\text{ is the fully antisymmetric tensor with }\varepsilon^{12\cdots n}=1\}.$$

Herewith in all these cases CRS of the (1.3*b*) type are fulfilled owing to initial CRS for a_i^+, a_j and syzygies of the invariant theory [1, 22]. From the mathematical viewpoint it is also of interest to examine non-Fock irreps of deformed algebras as well as new classes of orthogonal polynomials related to g_d. Specifically, one of practically important tasks here is in getting appropriate deformed analogues of the Rodrigues formula for concise representations of such polynomials. We also note that the scheme of section 2 can be generalized (via JS mapping) to the case of two or more interacting subsystems described in terms of generators of two or more usual Lie algebras. A simple example is provided by the point-like Dicke model given in terms of the $su(2)$ and $h(1)$ algebras [25].

As for developing physical aspects of the work we first of all point out approximate schemes of analyzing both energy and dynamic regimes which are based on using generalized HP mappings of the (3.15) type; specifically, one can determine in such a way interrelations between polynomially and q-deformed Lie algebras in order to display different exotic states and phenomena [25–29] in realistic multi-particle models. It is also of interest to examine possibilities of constructing evolution operators $U_H(t, t_0)$ associated with g_d along the line of the approach [39]. Another promising way here consists in representations of $U_H(t, t_0)$ in the form of power series in generators of g_d. For example, for the above algebras $sl_d(2)$ one can use the ansatz

$$U_H(t, t_0) = \sum_n \{A_n^H(Y_0; t, t_0)(Y_-)^n + (Y_+)^n B_n^H(Y_0; t, t_0)\} \tag{4.1}$$

where operator functions $A_n^H(\ldots)$ and $B_n^H(\ldots)$ are determined from some differential-difference equations [40].

Furthermore, the analysis of physical tasks, related to $sl_d(2)$ in the Heisenberg picture [40] leads to generalized Bloch equations whose quasiclassical solutions are expressed in terms of special Abelian functions [41]. Indeed, as follows from (2.7) and (3.2), the Heisenberg equations for collective dynamic variables $Y_\alpha(t)$, related to generators $Y_\alpha \in sl_d^{(n;p)}(2)$ are reduced to the form

$$i\hbar \mathrm{d}Y_0/\mathrm{d}t = bY_+ - b^* Y_- \tag{4.2a}$$

$$i\hbar \mathrm{d}Y_+/\mathrm{d}t = -aY_+ - b^*\varphi_{n,p}(Y_0),\ \varphi_{n,p}(Y_0) = \psi_{n,p}(Y_0+1) - \psi_{n,p}(Y_0) \tag{4.2b}$$

$$i\hbar \mathrm{d}Y_-/\mathrm{d}t = aY_- + b\varphi_{n,p}(Y_0) \tag{4.2c}$$

which coincide with the Bloch equations [42] in the case $n+p=2$ in $\psi_{n,p}(Y_0)$.

The solution of (4.2) is reduced to solving the only nonlinear equation

$$\mathrm{d}^2Y_0/\mathrm{d}t^2 = A\tilde{C} - \hbar^2A^2Y_0(t) + B\varphi_{n,p}(Y_0(t)) \tag{4.3}$$

where $A = a/\hbar^2$, $B = 2|b|^2/\hbar^2$, $\tilde{C} = H - C$ is an integral of motion. In the mean-field approximation, given by the relation

$$\langle \varphi_{n,p}(Y_0(t)) \rangle = \varphi_{n,p}(\langle Y_0(t) \rangle) \tag{4.4}$$

we get from (4.3) for the c-number function $y(t) = \langle Y_0(t) \rangle$ the equation

$$\left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 = 2\left[A\langle \tilde{C} \rangle y(t) - \frac{\hbar^2A^2}{2}(y(t))^2 + B\int^{y(t)} \mathrm{d}y\, \varphi_{n,p}(y) + D\right] \tag{4.5}$$

which is solved for polynomial functions $\varphi_{n,p}(x)$ in terms of hyperelliptic integrals defining special Abelian functions [41] (unlike the case of exponential functions $\varphi(y)$ for q-deformed algebras $sl_q(2)$). Therefore, exact operator solutions of (4.2) and (4.3), in a form like (4.1), seem to determine operator analogues of Abelian functions which, perhaps, are related to the problems of quantization on algebraic (Abelian) varieties [43, 41(b)]. It is also of interest to examine interrelations of these observations with possibilities of solving algebraic equations (3.7) in terms of theta-constants and hyperelliptic integrals [41(*b*)].

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Appendix. The three-boson model

In this appendix we consider applications of our approach to the three-boson model with the Hamiltonian

$$H=\sum_{i=1}^{3}\omega_i a_i^+ a_i + g a_1^+ a_2^+ a_3 + g^* a_1 a_2 a_3^+ \tag{A.1}$$

which is widespread in quantum optics and laser physics [44–46] and follows from (2.1) in the case $n=2$, $p=1$, $m_1=2$, $m_2=1$, $g^{j_1}_{i_1 i_2}=g\delta_{i_1 1}\delta_{i_2 2}\delta_{j_1 3}$.

In accordance with the general remarks of section 2 we find $G_{\mathrm{inv}}(H)=C_2\otimes U_0(1)\otimes U_1(1)$ where $U_0(1)=\exp(\mathrm{i}\lambda_0 R_0)$, $3R_0=(a_1^+a_1+2a_3^+a_3+a_2^+a_2)$, $U_1(1)=\exp(\mathrm{i}\lambda_1 R_1)$, $R_1=a_1^+a_1-a_2^+a_2$, and $g_d(H)=su_d^{(2,1)}(2)$ with generators

$$Y_0=\tfrac{1}{3}\left(\sum_{i=1}^{2}a_i^+a_i-a_3^+a_3\right)\qquad Y_+=a_1^+a_2^+a_3,\; Y_-=(Y_+)^+$$

and the structure polynomial

$$\psi_{2,1}(Y_0)\equiv\gamma(Y_0;R_0,R_1)=(R_0-Y_0+1)\left(Y_0+\frac{R_0+R_1}{2}\right)\left(Y_0+\frac{R_0-R_1}{2}\right). \tag{A.2}$$

Then, Hamiltonian (A.1) can be written in the form (3.2) with $a=\omega_1+\omega_2-\omega_3$, $b=g$, $C=R_0(\omega_1+\omega_2/2+\omega_3)+R_1(\omega_1-\omega_2)/2$, and the decomposition (2.6) takes the form

$$\begin{aligned}L_F(3)=\sum_{[l_1,l_2,l_3]}^{\oplus}L(l_1,l_2,l_3)&=\sum_{s=0}^{\infty}\Bigg\{L(-s/3,2s/3,0)\\&\quad+\sum_{k=1}^{\infty}\left[L\left(\frac{k-s}{3},\frac{k+2s}{3},k\right)+L\left(\frac{k-s}{3},\frac{k+2s}{3},-k\right)\right]\Bigg\}\\&=\sum\mathrm{Span}\{(Y_+)^r|[l_i]\rangle\}\end{aligned} \tag{A.3}$$

where

$$|[l_1, l_2, k]\rangle = [k!s!]^{-1/2}(a_1^+)^k(a_3^+)^s|0\rangle, \ |[l_1, l_2, -k]\rangle = [k!s!]^{-1/2}(a_2^+)^k(a_3^+)^s|0\rangle$$

$$k, s = 0, 1, 2, \ldots, Y_0|[l_i]\rangle = \left(l_1 = \frac{k-s}{3}\right)|[l_i]\rangle$$

$$R_0|[l_i]\rangle = \left(l_2 = \frac{k+2s}{3}\right)|[l_i]\rangle, R_1|[l_i]\rangle = (l_3 = \pm k)|[l_i]\rangle. \tag{A.4}$$

All of the spaces $L([l_i])$ are two-side bounded, i.e. they have highest vectors $|[l_i]; M\rangle = N(Y_+)^M|[l_i]\rangle$, $M = s$, $Y_+|[l_i]; M\rangle = 0$, and finite dimensions $d([l_i]) = s + 1$.

The substitution of $\psi_{2,1}(l_1 + f) \equiv \gamma(l_1 + f; l_2, l_3)$ from (A.2) into (3.4) and (3.7) leads, after some algebra, to the equations

$$(af - \lambda)Q_f + gQ_{f-1} + g^*(s-f)(f+1)(f+k+1)Q_{f+1} = 0, f = 0, 1, \ldots, s \tag{A.5}$$

where $a = \omega_1 + \omega_2 - \omega_3$, $\lambda = E - s\omega_3 - k\omega_1$ for $l_3 = k$ and $\lambda = E - s\omega_3 - k\omega_2$ for $l_3 = -k$, and boundary conditions

$$Q_{-1} = 0, (Q_{s+1} = 0) \Rightarrow (sa - \lambda)Q_s + gQ_{s-1} = 0 \tag{A.6}$$

for solving the spectral problem with the Hamiltonian (A.1) on the $(s+1)$-dimensional subspaces $L((k-s)/3, (k+2s)/3, \pm k)$.

From (A.5) and (A.6) we find

$$Q_f = Q_0 P_f(\lambda)/(g^*)^f (s)^{(f)} f!(k+f)^{(f)} \qquad f = 1, \ldots, s, (a)^{(b)} = a!/(a-b)! \tag{A.7}$$

where $P_f(\lambda)$ is a polynomial of the fth order in λ, and the spectra $\{\lambda_\alpha\}$, $\{E_\alpha\}$ of $(s+1)$ admissible values λ_α, E_α are determined by solving the algebraic equation

$$P_{s+1}(\lambda) = 0 \Leftarrow ((sa - \lambda)Q_s + gQ_{s-1} = 0). \tag{A.8}$$

For lowest values of s one can find the exact form of $P_f(\lambda)$ and roots of (A.8) by hand; for example, for $s = 1$ we have

$$P_0(\lambda) = 1 \qquad P_1(\lambda) = \lambda$$

and

$$P_2(\lambda) = 0 = \lambda(\lambda - a) - |g|^2(k+1), \lambda_\pm = \frac{a}{2} \pm \left(\frac{a^2}{4} + |g|^2(k+1)\right)^{1/2}.$$

For higher values of s these tasks are easily solved using simple computer routines; specifically, for solving (A.8) one can use the Umemura algorithm [41(*b*)] expressing roots of algebraic equations in terms of theta-constants.

We note that the above algorithm of solving the model (A.1), given by equations (3.3), (A.3)–(A.8), is simpler than that obtained by using the algebraic Bethe Ansatz [46]. Specifically, it requires only the determination of spectral parameter λ (from solving (A.8)) instead of many E_i (from solving a system of nonlinear algebraic equations) in [46]. Besides, our approach takes into account explicitly symmetries of the Hamiltonian (A.1) that allows us to give, unlike [46], a clear physical classification (see, e.g., equations (A.3), (A.4)) of the (A.1) eigenstates $|E\rangle \in L_F(3)$; specifically, we have $k = |N_1 - N_2|$, $s = N_3 + (N_1 + N_2 - |N_1 - N_2|)/2$ where N_i stands for the population of ith mode. These remarks are in accordance with results of similar comparisons (but without using a Lie-algebraic formulation) in [47] for the quantum

Ablowitz–Ladik system which is related to a deformed oscillator algebra. (The author thanks Professor A J Macfarlane for a stimulating discussion of this point.)

As a final remark we point out that the substitution of (A.2) in (4.2), (4.3) and (4.5) enables one automatically to reproduce the results of [44] where elliptic functions were first introduced for analysing the model (A.1). A comparison of our approach with formal and complicated solutions [45] requires further investigations.

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